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CHAIN PROPERTIES IN Pω

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0. Introduction

In the past decade, semantical domains of programming languages have been modeled by c.p.o.'s ([5, 6]), i.e. partially ordered sets where every ascending chain has a least upper bound. As well important are chain-complete subsets of c.p.o.'s which also have the "closed under chain" property. This paper is devoted to the study of chain-complete subsets of the lattice Pω, a universal domain for data types ([6, 8]).

A subset E of a c.p.o. is said to be chain-complete if it contains the least upper bound of every ascending chain lying in it (i.e. E). Let us first mention how chain-complete sets arise in mathematical semantics:

(1) Suppose we model data types by continuous lattices [5] and we want to know when one data type is "contained" or embedded in another. Mathematically, when does a subset E of a given continuous lattice D become a continuous lattice under the induced ordering? A sufficient condition is that E is a retract of D ([5]). Retractions from D to D are continuous mappings satisfying $r \circ r = r$. A retract of D is given by the range of some retraction map $r: D \to D$. Since r is idempotent, it can be verified that the range $r(D)$ consists of all the fixed points of r, i.e.

$$r(D) = \{x \in D \mid r(x) = x\}.$$

Note that $r(D)$ is a chain-complete subset of D. Thus a necessary condition for E to be a retract of a given data type D is that E is a chain-complete subset of D.

(2) Semantics of programming languages can be given in various ways. For example in [3], the semantics of a simple programming language was given using the direct method and the continuation method. To relate the two methods of language descriptions, Reynolds [3] defined a relation $r \subseteq D_1 \times D_2$ such that

$$r(\mathcal{M}_d[\![e]\!], \mathcal{M}_c[\![e]\!])$$

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holds for all expressions e where

(i) $\mathcal{M}_d$ and $\mathcal{M}_c$ stand for the valuation mappings for the direct method and the continuation method respectively;

(ii) D_1 is the semantical domain for expressions in the direct method and D_2 is the semantical domain for expressions in the continuation method.

Since r is in general defined recursively (e.g. when D_1 and D_2 are reflexive domains), the difficulty is to show that such an r ever exists! In most cases, the definition of r is not even continuous with respect to its components and one can hardly appeal to Tarski's fixedpoint theorem. This disaster disappears if one works with directed-complete relations [3]; a relation $r \subseteq D_1 \times D_2$ is directed complete if it is a chain-complete subset of $D_1 \times D_2$. A "least" relation does exist if it is defined in terms of directed-complete subrelations! Fortunately, the relations that creep up in relating two semantical domains of programming languages are often directed-complete.

(3) Admissible predicates were introduced in [2] for de Bakker–Scott's computational induction rule. Roughly speaking, a predicate $\varphi(\cdot)$ is admissible if it has the "continuous" property that for every continuous functional t, if $\varphi(t^i(\Omega))$ holds for all $i \in \omega$, then $\varphi(\mathrm{lub}\{t^i(\Omega)\})$ holds. Chain-complete sets are the semantical counterparts of the above admissible predicates, i.e. each admissible predicate defines a chain-complete subset of a corresponding semantical domain.

We have seen that chain-complete sets arise in mathematical semantics, language descriptions and program verification. Our main interests in this paper lie in chain-complete sets that can be defined by admissible predicates, since non-definable sets hardly have any significance in computer science. Among the admissible predicates, there are those of the form

$$\varphi(f)\colon \alpha[f] \equiv \beta[f]$$

where α and β are continuous functionals ([2]). The above admissible predicates have the characteristics that they can be expressed as equations with continuous functionals on both sides. As was shown in [2], these predicates arise very frequently in program verification. The semantical counterparts of these predicates are chain-complete sets of the form

$$\{x \in D \mid f(x) = g(x)\}$$

where f, g are continuous functions from D to D. The above chain-complete sets are identified with the $\mathcal{B}_\delta$ sets in [7]. We will study properties of these sets in detail in Section 2.

In this paper, we base our studies of chain-complete sets in the lattice Pω. Some general facts about Pω are given in Section 1. Among Scott's continuous lattices [5], there are the effectively given ones where we can formulate the notion of a computable element. When we describe the denotation semantics of a programming language, we only use effectively given domains since we want to say that the meanings of commands or expressions are computable elements in the corresponding semantical domains. Effectively given lattices are carefully axiomatized in [8] where they are called admissible domains. Our main result of [8] is that the computable retracts of Pω give us (modulo isomorphism) all the effectively given continuous lattices. Thus if we model data types by continuous lattices, then Pω deserves to be called a universal domain for data types. By choosing to study chain-complete sets in the universal domain Pω, we can obtain results that can be generalized to most domains.

Our main theorem of this paper is a characterization theorem for the class of chain-complete sets whose complements are also chain-complete. We show that E and E^c are chain-complete if and only if they can be expressed in the form $\{x \in \mathrm{P}\omega \mid f(x) = g(x)\}$ which we discussed earlier. The nontrivial direction is $\Rightarrow$, because it goes from chain properties to definability properties.

Let us describe an application of our main theorem. When we prove congruence of two semantical descriptions of a programming language, we used to come up with predicates or relations which we want to be directed-complete. As we mentioned earlier, working with directed-complete relations would guarantee us the existence of a recursively defined relation. The question is: how do we test if a specified relation is directed-complete? It is easy to see that chain-complete sets are closed under finite unions and arbitrary intersections. In general, they are not closed under complements. Now suppose we come up with a relation $r(x, y)$ which has this following form:

$$r(x, y)\colon (\forall z)[E_1(z, x, y) \supset E_2(z, x, y)]$$

where E_1 and E_2 are chain-complete sets. Since $E_1(z, x, y) \supset E_2(z, x, y)$ is equivalent to $\neg E_1(z, x, y) \vee E_2(z, x, y)$, it is not in general true that r is a directed-complete relation. A sufficient condition for r to be directed-complete is that the complement E_1^c of E_1 is also chain-complete. Now in Section 3, we describe Hausdorff difference hierarchy, a transfinite hierarchy giving us all the $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ sets in Pω, i.e. $\mathscr{B}_\delta$ sets whose complements are also $\mathscr{B}_\delta$. By our main theorem and the normal form theorem in [7], these $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ sets consist of all chain-complete sets whose complements are also chain-complete. Thus using our knowledge of how the sets in the Hausdorff hierarchy are generated, we can test if E_1 and E_1^c are both chain-complete.

An important subclass of the $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ sets is given by the $\mathscr{B}$ sets. Most of the $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ sets appearing in language descriptions are $\mathscr{B}$ sets. In Section 6, we give a criteria to determine when a given set is a $\mathscr{B}$ set.

Further applications of our theorems on chain properties are given in Sections 7 and 8. Throughout this paper, $\{e_i\}_{i\in\omega}$ stands for the effective enumeration of all finite sets of integers as in [4].

1. Some basic facts about Pω

The elements of Pω are subsets of integers. Set theoretically, Pω and Cantor set are the same. The essential difference between Pω and Cantor space is the topology.

The open sets in Pω are generated by the following sub-base: the class of $\{x \mid n \in x\}$, whereas the open sets in Cantor space are generated by a larger sub-base: the class of $\{x \mid n \in x\}$ and $\{x \mid n \notin x\}$. Thus the open sets in Cantor space can be specified by positive or negative information, but the open sets in Pω are strictly specified by positive information.

A more straightforward way to specify the topology of Pω is as follows: a subset E of Pω is open if

(i) whenever $x \in E$ and $x \subseteq y$, then $y \in E$, i.e. E is closed under "superset"; and
(ii) whenever $x \in E$, then some finite subset of x is in E.

By (i), every non-empty open set contains the element ω. Hence every pair of non-empty open sets in Pω has a non-empty intersection, and therefore Pω is not Hausdorff. A further consequence is that Pω is not metrizable simply because metric spaces are always Hausdorff.

A T_0 space is a space where the elements are uniquely determined by their neighborhoods. The topology of Pω turns Pω to a T_0 space. As a T_0 space, Pω has the "universal" property that every separable T_0 space can be embedded in it.

For suppose D is a T_0 space with a countable base $\{\mathcal{U}_n\}_{n\in\omega}$ for the topology, define the map $i: D \to$ Pω as follows:

$$i(x) = \{n \mid x \in \mathcal{U}_n\}.$$

It is routine to check that i embeds D in Pω.

If we partially order Pω by set inclusion, then Pω becomes a lattice, in fact an algebraic continuous lattice [5] where the isolated elements are all the finite sets of integers. A retract D of Pω is given by a retraction map $r:\mathrm{P}\omega \to \mathrm{P}\omega$ which is continuous and idempotent, i.e. $r \circ r = r$. The range of r constitutes the elements in D; it can be shown that the set D coincides with the set of all fixed points of r. It is proved in [5] that every retract of Pω is a separable continuous lattice where the partial ordering is given by the induced ordering from Pω.

Conversely, if D is a separable continuous lattice with a countable base $\{\mathcal{U}_n\}_{n\in\omega}$, then the following map $r:\mathrm{P}\omega \to \mathrm{P}\omega$,

$$r(x) = \{m \mid \bigsqcup\{\sqcap \mathcal{U}_n \mid n \in x\} \in \mathcal{U}_m\}$$

retracts Pω into $r(\mathrm{P}\omega)$. D can be shown to be isomorphic to $r(\mathrm{P}\omega)$, hence D is a retract of Pω. In summary, the retracts of Pω give us all the separable continuous lattices (modulo isomorphism).

2. $\mathcal{B}_\sigma$ and $\mathcal{B}_\delta$ sets

Given a subset E of Pω, we say that E is closed under chain (or chain-complete) if $\bigcup_{i\in\omega} x_i \in E$ whenever $\{x_i\}_{i\in\omega}$ is an ascending chain in E. In this section, we define a class of $\mathcal{B}_\delta$ sets in Pω which all have the "closed under chain" property. It was shown in [7] that these $\mathcal{B}_\delta$ sets coincide with the subsets of the form $\{x \mid f(x) = g(x)\}$ where f, g range over continuous functions from $P\omega$ to $P\omega$. In this sense, the $\mathcal{B}_\delta$ sets represent the semantical counterparts of admissible predicates [2] expressible in the form $\alpha[f] \equiv \beta[f]$. Important examples of $\mathcal{B}_\delta$ sets are retracts of $P\omega$ because retracts are of the form $\{x \mid f(x) = x\}$ where f range over the retraction maps.

Definition. (i) E is a $\mathcal{B}$ set if it is some Boolean combination of open sets;

(ii) E is a $\mathcal{B}_\delta$ set if it is some countable intersection of $\mathcal{B}$ sets; and

(iii) E is a $\mathcal{B}_\sigma$ set if E^c (complement of E) is $\mathcal{B}_\delta$ — equivalently, E is a countable union of $\mathcal{B}$ sets.

We say that a set is $\mathcal{I} \dot{\cap} \mathcal{S}$ if it is given by the intersection of some open set and some closed set. We observed in [7] that the $\mathcal{B}$ sets in $P\omega$ are all finite unions of $\mathcal{I} \dot{\cap} \mathcal{S}$ sets, and the $\mathcal{B}_\sigma$ sets are countable unions of $\mathcal{I} \dot{\cap} \mathcal{S}$ sets.

Given $y \in P\omega$, let us consider the complexity of the singleton subset $\{y\}$. Note:

$$\{y\} = \bigcap \{\{x \mid e \subseteq x\} \cap \{x \mid x \subseteq y\} \mid e \text{ is a finite subset of } y\}.$$

Since each $\{x \mid e \subseteq x\} \cap \{x \mid x \subseteq y\}$ is a $\mathcal{I} \dot{\cap} \mathcal{S}$ set, the set $\{y\}$ is $\mathcal{B}_\delta$. When y is a finite subset of integers, then

$$\{y\} = \{x \mid y \subseteq x\} \cap \{x \mid x \subseteq y\}$$

which therefore is $\mathcal{I} \dot{\cap} \mathcal{S}$, and therefore a $\mathcal{B}$ set.

It can be easily verified that chain-complete sets are closed under finite unions and arbitrary intersections. The open sets and the closed sets in $P\omega$ are all closed under chain. Since the $\mathcal{B}$ sets in $P\omega$ are finite unions of $\mathcal{I} \dot{\cap} \mathcal{S}$ sets, the $\mathcal{B}$ sets are chain-complete. And since the $\mathcal{B}_\delta$ sets are countable intersections of $\mathcal{B}$ sets, the $\mathcal{B}_\delta$ sets are therefore chain-complete.

While the $\mathcal{B}_\delta$ sets enjoy the "closed under chain" property, their complements have the following "finitary" property: E has *finitary* property if for every x in E, there exists some finite subset e of x such that the set $\{y \mid e \subseteq y \subseteq x\}$ is contained in E. Not surprisingly, the "closed under chain property" and the "finitary property" are "almost" dual notions of each other, as we shall verify in Section 5. The next few propositions serve to clarify the properties of $\mathcal{B}_\sigma$ and $\mathcal{B}_\delta$ sets in more detail.

Proposition 1. (a) *E has finitary property if and only if E is some union of $\mathcal{I} \dot{\cap} \mathcal{S}$ sets. Hence all $\mathcal{B}_\sigma$ sets have finitary property.*

(b) *Every non-empty subset of* $P\omega$ *with finitary property contains some finite set of integers. Hence every non-empty $\mathcal{B}_\sigma$ set contains some finite set of integers.*

(c) *There are sets which have finitary property but not $\mathcal{B}_\sigma$.*

(d) *$\mathcal{B}_\delta$ sets do not necessarily have finitary property.*

Proof. (a) ($\Rightarrow$) Use the fact that for any subset x of ω and finite subset e of x, the set $\{y \mid e \subseteq y \subseteq x\}$ is $\mathcal{J}\dot{\cap}\mathcal{S}$. ($\Leftarrow$) Note that all $\mathcal{J}\dot{\cap}\mathcal{S}$ sets have finitary property, and the class of subsets in $P\omega$ with finitary property is closed under arbitrary union.

(b) This follows trivially from the definition of finitary property.

(c) Let E be $\{x \mid x$ is co-infinite$\}$. E has finitary property but E is not $\mathcal{B}_\sigma$.[1]

(d) Let E be $\{x \mid x$ is infinite$\}$. E is $\mathcal{B}_\delta$ but E does not contain any finite set of integers.

Proposition 2. (a) *There are sets which are closed under chain but not* $\mathcal{B}_\delta$.

(b) $\mathcal{B}_\sigma$ *sets are not necessarily closed under chain.*

Proof. (a) Let E be $\{x \mid x$ is co-finite$\}$. E is closed under chain but not $\mathcal{B}_\delta$.

(b) Let E be $\{x \mid x$ is finite$\}$. E is $\mathcal{B}_\sigma$ but not closed under chain.

Proposition 3. (a) *Suppose E is $\mathcal{B}_\delta$ but not $\mathcal{B}_\sigma$, then E contains some infinite subset of ω.*

(b) *Suppose E is $\mathcal{B}_\sigma$ and y is some infinite subset of ω in E. If we remove y from E, the resulting set $E' = E\backslash\{y\}$ is not $\mathcal{B}_\delta$.*

Proof. (a) If the given $\mathcal{B}_\delta$ set E does not contain any infinite subset of ω, then E would be some union of singleton sets $\{e_i\}$, hence is $\mathcal{B}_\sigma$ (we noted earlier that the singleton set $\{e\}$ is $\mathcal{J}\dot{\cap}\mathcal{S}$ whenever e is finite).

(b) Suppose y is some infinite subset of ω in the given $\mathcal{B}_\sigma$ set E. Since E has finitary property, there exists some finite subset e of y such that the set $\{x \mid e \subseteq x \subseteq y\}$ is contained in E. Now the set $\{x \mid e \subseteq x \subsetneq y\}$ is contained in E', and clearly E' is not closed under chain, hence not $\mathcal{B}_\delta$.

The "closed under chain property" can be phrased in terms of directed sets. A subset E of $P\omega$ is directed if for every pair y_1, y_2 in E, there exists some upper bound z of y_1 and y_2 in E. We say that E is directed-complete [3] if $\bigcup\{x \mid x \in E'\} \in E$ whenever E' is a directed subset of E. We leave the reader to verify that E is chain-complete if and only if E is directed-complete.

In the next two sections, we study $\mathcal{B}_\sigma \cap \mathcal{B}_\delta$ sets in $P\omega$, or sets that are both $\mathcal{B}_\sigma$ and $\mathcal{B}_\delta$. The inclusions between the various classes of sets is given as follows:

$$\mathcal{B} \subsetneq \mathcal{B}_\sigma \cap \mathcal{B}_\delta \begin{array}{l} \subsetneq \mathcal{B}_\sigma \\ \subsetneq \mathcal{B}_\delta \end{array}$$

The singleton set $\{\omega\}$ is $\mathcal{B}_\delta$ but not $\mathcal{B}_\sigma$ since its complement $\{\omega\}^c$ is not chain-complete; this explains why the class of $\mathcal{B}_\sigma \cap \mathcal{B}_\sigma$ sets is a proper subclass of the $\mathcal{B}_\delta$ sets or the $\mathcal{B}_\sigma$ sets. We will show in Section 6 why the $\mathcal{B}$ sets form a proper subclass of the $\mathcal{B}_\sigma \cap \mathcal{B}_\delta$ sets.

[1] It is well-known that E is not a $\mathcal{J}_\sigma$ set in Cantor space [4]. Since the $\mathcal{B}_\sigma$ sets $P\omega$ form a subclass of the $\mathcal{J}_\sigma$ sets in Cantor space [7], E is not a $\mathcal{B}_\sigma$ set.

3. Hausdorff difference hierarchy

In this section, we introduce a transfinite hierarchy called the Hausdorff difference hierarchy and show that the sets in the hierarchy give us all the $\mathscr{B}_\sigma \cap \mathscr{B}_\sigma$ sets in Pω.

Suppose E is a subset of Pω, Hausdorff sequences $\{P^E_\alpha\}_{\alpha\in \mathrm{Or}}$ and $\{Q^E_\alpha\}_{\alpha\in \mathrm{Or}}$ are defined as follows: for $\alpha \in \mathrm{Or}$ (the class of ordinals),

$$P^E_0 = E, \qquad P^E_\alpha = \mathrm{Cl}(Q^E_\alpha) \cap E,$$
$$Q^E_0 = E^c, \qquad Q^E_{\alpha+1} = \mathrm{Cl}(P^E_\alpha) \cap E^c,$$

for limit ordinal α,

$$P^E_\alpha = \mathrm{Cl}(Q^E_\alpha) \cap E,$$
$$Q^E_\alpha = \bigcap_{\beta<\alpha} Q^E_\beta.$$

(Cl stands for the closure operator in a topological space.)

Here are some of the properties of $\{P^E_\alpha\}_{\alpha\in \mathrm{Or}}$ and $\{Q^E_\alpha\}_{\alpha\in \mathrm{Or}}$:

(i) $\{P^E_\alpha\}_{\alpha\in \mathrm{Or}}$ and $\{Q^E_\alpha\}_{\alpha\in \mathrm{Or}}$ are decreasing sequences.

(ii) For $\alpha \in \mathrm{Or}$, $\mathrm{Cl}(P^E_\alpha) = P^E_\alpha \cup Q^E_{\alpha+1}$ and $\mathrm{Cl}(Q^E_{\alpha+1}) = P^E_{\alpha+1} \cup Q^E_{\alpha+1}$. Also note $P^E_0 \cup Q^E_0 = P\omega$ and $P^E_\alpha \cup Q^E_\alpha$ is a closed set for all limit ordinals α. Thus $\{P^E_\alpha \cup Q^E_\alpha\}_{\alpha\in \mathrm{Or}}$ and $\{P^E_\alpha \cup Q^E_{\alpha+1}\}_{\alpha\in \mathrm{Or}}$ are decreasing sequences of closed sets in Pω.

(iii) For $\alpha \in \mathrm{Or}$,

$$P^E_\alpha \backslash P^E_{\alpha+1} = \mathrm{Cl}(P^E_\alpha) \backslash \mathrm{Cl}(Q^E_{\alpha+1})$$

and

$$Q^E_{\alpha+1} \backslash Q^E_{\alpha+2} = \mathrm{Cl}(Q^E_{\alpha+1}) \backslash \mathrm{Cl}(P^E_{\alpha+1}).$$

Also note $Q^E_0 \backslash Q^E_1 = [\mathrm{Cl}(P^E_0)]^c$ and

$$Q^E_\alpha \backslash Q^E_{\alpha+1} = (P^E_\alpha \cup Q^E_\alpha) \backslash \mathrm{Cl}(P^E_\alpha)$$

for all limit ordinals α. Thus for all $\alpha \in \mathrm{Or}$, $P^E_\alpha \backslash P^E_{\alpha+1}$ and $Q^E_\alpha \backslash Q^E_{\alpha+1}$ are differences of two closed sets, hence they are $\mathscr{J} \cap \mathscr{S}$ sets in Pω.

(iv) For $\delta \in \mathrm{Or}$, say that P^E_δ stabilizes if for all $\delta < \beta$, $P^E_\delta = P^E_\beta$; similarly, say that Q^E_δ stabilizes if for all $\delta < \beta$, $Q^E_\delta = Q^E_\beta$. It is well known that no uncountable strictly decreasing sequence of closed sets can exist in a separable space, in particular, Pω. As $\{P^E_\alpha \cup Q^E_\alpha\}_{\alpha\in \mathrm{Or}}$ is an uncountable decreasing sequence of closed sets, there exists $\delta, \gamma < \Omega$ (Ω is the first uncountable ordinal) such that P^E_δ and Q^E_γ stabilize. The set $P^E_\delta \cup Q^E_\gamma$ is called the *residue of E*. Note:

$$\begin{aligned} E &= [P^E_0 \backslash P^E_1] \cup \cdots \cup [P^E_\alpha \backslash P^E_{\alpha+1}] \cup \cdots \cup P^E_\delta \\ &= [\mathrm{Cl}(P^E_0) \backslash \mathrm{Cl}(Q^E_1)] \cup \cdots \cup [\mathrm{Cl}(P^E_\alpha) \backslash \mathrm{Cl}(Q^E_{\alpha+1})] \cup \cdots \cup P^E_\delta \end{aligned}$$

and

$$E^c = [Q_0^E \backslash Q_1^E] \cup \cdots \cup [Q_{\alpha+1}^E \backslash Q_{\alpha+2}^E] \cup \cdots \cup Q_\gamma^E$$
$$= [\mathrm{P}\omega \backslash \mathrm{Cl}(P_0^E)] \cup \cdots \cup [\mathrm{Cl}(Q_{\alpha+1}^E) \backslash \mathrm{Cl}(P_{\alpha+1}^E)] \cup \cdots \cup Q_\gamma^E.$$

Hausdorff hierarchy $\{\mathscr{H}_\alpha\}_{\alpha<\Omega}$ in Pω is defined as follows: for $\lambda = 0$ or any limit ordinal $< \Omega$, and $n \in \omega$,

$$\mathscr{H}_{\lambda+2n} = \{E \mid Q_{\lambda+n+1}^E \text{ stabilizes}\},$$
$$\mathscr{H}_{\lambda+2n+1} = \{E \mid P_{\lambda+n+1}^E \text{ stabilizes}\}.$$

Thus $\{\mathscr{H}_\alpha\}_{\alpha<\Omega}$ classifies the subsets E of Pω in terms of the ordinal $\delta < \Omega$ at which P_δ^E or Q_δ^E stabilizes. Let us note that $\{\mathscr{H}_\alpha\}_{\alpha<\Omega}$ is an increasing hierarchy.

Hausdorff difference hierarchy $\{\mathscr{D}_\alpha\}_{\alpha<\Omega}$ is defined to be a subhierarchy of $\{\mathscr{H}_\alpha\}_{\alpha<\Omega}$, consisting of all the subsets of Pω with an empty residue. The formal definition is as follows: for $\lambda = 0$ or any limit ordinal $\lambda < \Omega$, and $n \in \omega$,

$$\mathscr{D}_{\lambda+2n} = \{E \mid Q_{\lambda+n+1}^E = \emptyset\},$$
$$\mathscr{D}_{\lambda+2n+1} = \{E \mid P_{\lambda+n+1}^E = \emptyset\}.$$

Not every subset E of Pω has a vanishing residue. For let E be $\{\omega\}$. Then for all $\alpha \in$ Or, $P_\alpha^E = E$ and $Q_\alpha^E = E^c$. Hence P_0^E and Q_0^E stabilize, and the residue of E is the whole set Pω.

The sets in $\{\mathscr{D}_n\}_{n<\omega}$ have the property that their residues vanish at a finite stage. These sets turn out to coincide with all the $\mathscr{B}$ sets in Pω (see [7]). For the rest of this section, we shall demonstrate that the sets in $\{D_\alpha\}_{\alpha<\Omega}$ give us all the $\mathscr{B}_\sigma \cap \mathscr{B}_\sigma$ sets in Pω.

Proposition 4. *The sets in* $\{\mathscr{D}_\alpha\}_{\alpha<\Omega}$ *are* $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ *sets in* Pω.

Proof. Suppose E is in $\{\mathscr{D}_\alpha\}_{\alpha<\Omega}$ and $P_\delta^E = Q_\delta^E = \emptyset$. We can express E and E^c as follows:

$$E = [\mathrm{Cl}(P_0^E) \backslash \mathrm{Cl}(Q_1^E)] \cup \cdots \cup [\mathrm{Cl}(P_\alpha^E) \backslash \mathrm{Cl}(Q_{\alpha+1}^E)] \cup \cdots$$

and

$$E^c = [\mathrm{P}\omega / \mathrm{Cl}(P_0^E)] \cup \cdots \cup [\mathrm{Cl}(Q_{\alpha+1}^E) \backslash \mathrm{Cl}(P_{\alpha+1}^E)] \cup \cdots$$

where $\alpha + 1 < \delta$.

Since E and E^c are countable unions of $\mathscr{J} \cap \mathscr{S}$ sets, they are $\mathscr{B}_\sigma$; hence E is a $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ set in Pω.

To see that every $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ set E is in $\{\mathscr{D}_\alpha\}_{\alpha<\Omega}$ (that is, E has an empty residue), we need the following two lemmas.

Lemma 1. *Say that E admits an infinite alternating chain $\{x_i\}_{i\in\omega}$ if $\{x_i\}_{i\in\omega}$ is an increasing chain in* Pω *and for $i\in\omega$, $x_{2i}\in E$ and $x_{2i+1}\in E^c$. No $\mathscr{B}_\sigma\cap\mathscr{B}_\delta$ set admits an infinite alternating chain.*

Proof. Suppose E admits an infinite alternating chain $\{x_i\}_{i\in\omega}$. As E is a $\mathscr{B}_\sigma\cap\mathscr{B}_\delta$ set, E and E^c are both closed under chain, hence $x=\bigcup\{x_{2i}\mid i\in\omega\}=\bigcup\{x_{2i+1}\mid i\in\omega\}$ has to belong to $E\cap E^c$, contradiction.

Lemma 2. (a) *Let E be any subset of* Pω. *Then for every finite subset e of ω in* $\mathrm{Cl}(E)\backslash E$, *there exists some $y\in E$ containing e.*

(b) *When the set E is $\mathscr{B}_\sigma$, then y can be taken to be a finite set of integers.*

Proof. (a) Suppose e is a finite set of integers in $\mathrm{Cl}(E)\backslash E$. Consider the set $E'=\mathrm{Cl}(E)\backslash\{x\mid e\subseteq x\}$ which is closed. E cannot be a subset of E', for otherwise E' is a closed set (smaller than $\mathrm{Cl}(E)$) containing E. Hence we can find some y in $E\backslash E'$; any such y contains e.

(b) Now assume E is $\mathscr{B}_\sigma$, then E has finitary property. Thus there exists some finite subset e' of y such that the set $\{x\mid e'\subseteq x\subseteq y\}$ is contained in E. Since e is a finite subset of y, we can find some finite subset e'' of y in E such that $e\subseteq e''$ holds.

Proposition 5. *Suppose E is a $\mathscr{B}_\sigma\cap\mathscr{B}_\delta$ set, then E has an empty residue.*

Proof. Suppose E is $\mathscr{B}_\sigma\cap\mathscr{B}_\delta$ and the residue of E, $P_\delta^E\cup Q_\gamma^E$, (where we assume that P_δ^E and Q_γ^E stabilize) is non-empty. Note that $P_\delta^E\cup Q_\gamma^E$ is a closed set, and that $P_\delta^E=(P_\delta^E\cup Q_\gamma^E)\cap E$ and $Q_\gamma^E=(P_\delta^E\cup Q_\gamma^E)\cap E^c$, both being an intersection of a closed set and a $\mathscr{B}_\sigma$ set, are non-empty $\mathscr{B}_\sigma$ sets in Pω. We shall derive a contradiction by showing that E admits an infinite alternating chain. Using induction, we define a sequence $\{x_i\}_{i\in\omega}$ of finite sets of integers as follows:

$i=0$. Since P_δ^E is a non-empty $\mathscr{B}_\sigma$ set, there exists some finite subset of ω in P_δ^E, let it be x_0.

Now assume x_i has been defined and is a finite subset of ω.

Case 1 (i is even). Apply Lemma 2 to obtain some finite subset x_{i+1} of ω in Q_γ^E such that x_i is contained in x_{i+1}.

Case 2 (i is odd). Again, apply Lemma 2 to obtain some finite subset x_{i+1} of ω in P_δ^E such that x_i is contained in x_{i+1}. Clearly E admits the infinite alternating chain $\{x_i\}_{i\in\omega}$, hence contradicting Lemma 1.

Combining Propositions 4 and 5, we conclude:

Theorem 1. *The sets in $\{\mathscr{D}_\alpha\}_{\alpha<\Omega}$ are precisely all the $\mathscr{B}_\sigma\cap\mathscr{B}_\delta$ sets in* Pω.

We close this section by observing a property of $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ sets which neither the $\mathscr{B}_\sigma$ sets nor the $\mathscr{B}_\delta$ sets share, namely: every $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ set is determined by the collection of finite sets of integers contained in it.

Proposition 6. *Let E, E' be $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ sets. Then $E = E'$ if and only if for all $i \in \omega$, $e_i \in E \leftrightarrow e_i \in E'$.*

Proof. ($\Rightarrow$) Trivial.

($\Leftarrow$) It suffices to show that E is contained in E'. Suppose y is in E. Because E has finitary property, we can find some finite subset e_k of y such that the set $\{x \mid e_k \subseteq x \subseteq y\}$ is contained in E. By our assumption, the directed set $\{e_i \mid e_k \subseteq e_i \subseteq y\}$ is also contained in E'. Since E' is closed under chain, the lub of $\{e_i \mid e_k \subseteq e_i \subseteq y\}$, namely y, is in E'.

4. A characterization of $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ sets

When E is a $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ set, then E and its complement E^c are both closed under chain. Our main theorem in this paper is that the converse also holds.

Theorem 2. *E is $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ if and only if E and E^c* are closed under chain.

Proof. ($\Rightarrow$) Trivial.

($\Leftarrow$) Suppose E and E^c are both closed under chain and E is not $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$. By Theorem 1, the residue of E, say $P_\delta^E \cup Q_\gamma^E$, is non-empty. Using induction, we define a sequence $\{x_i\}_{i\in\omega}$ of finite subsets of ω alternating in and out of $P_\delta^E \cup Q_\gamma^E$.

$i = 0$. Since $P_\delta^E \cup Q_\gamma^E$ is a non-empty closed set in $\mathrm{P}\omega$, $\emptyset$ belongs to $P_\delta^E \cup Q_\gamma^E$, say P_δ^E. Let x_0 be $\emptyset$.

Assume x_i has been defined and is a finite subset of ω.

Case 1 (i is even and $x_i \in P_\delta^E$).

Claim. There exists some finite subset x_{i+1} of ω in Q_γ^E such that x_i is contained in x_{i+1}.

Proof. As x_i belongs to $\mathrm{Cl}(Q_\gamma^E)$, we know by Lemma 2 that there exists some y in Q_γ^E such that x_i is contained in y. If y is a finite set of integers, take x_{i+1} to be y and we are done. So assume not. Note:

$$y = \bigcup_{k\in\omega} \{e_k \mid x_i \subseteq e_k \subseteq y\}.$$

The set $\{e_k \mid x_i \subseteq e_k \subseteq y\}$ is contained in $P_\delta^E \cup Q_\gamma^E$ because $P_\delta^E \cup Q_\gamma^E$ is a closed set and y is in it. If the set $\{e_k \mid x_i \subseteq e_k \subseteq y\}$ is contained in P_δ^E, then y would be in P_δ^E since P_δ^E is chain-complete (being the intersection of two chain-complete sets, namely, $P_\delta^E \cup Q_\delta^E$ and E). Since y is in Q_γ^E, we conclude that $Q_\gamma^E \cap \{e_k \mid x_i \subseteq e_k \subseteq y\}$ is non-empty. Take x_{i+1} to be any e_k in it.

Case 2. (i is odd and $x_i \in Q_\gamma^E$). Use the same idea in Case 1 to obtain some finite subset x_{i+1} of ω in P_δ^E such that x_i is contained in x_{i+1}.

We have constructed an increasing chain $\{x_i\}_{i\in\omega}$ of finite subsets of ω alternating in and out of E. Now $\{x_{2i}\}_{i\in\omega}$ and $\{x_{2i+1}\}_{i\in\omega}$ are increasing chains in E and E^c respectively, and they have the same lub, call it x. Since E and E^c are both closed under chain, x has to belong to $E \cap E^c$, contradiction.

5. More about chain properties

We remarked in Section 2 that the "closed under chain" property and the "finitary" property are almost dual to each other. To clarify this remark, we introduce some more notions in this section, namely:

(i) "closed under finite chain", which is weaker than "closed under chain";
(ii) "closed under strong chain", which is stronger than "closed under chain"; and
(iii) "semi-finitary" property, which is weaker than "finitary" property.

We shall prove the following two theorems:

(1) E is closed under finite chain if and only if E^c has semi-finitary property; and
(2) E is closed under strong chain if and only if E^c has finitary property.

E is *closed under finite chain* if for every increasing chain $\{x_i\}_{i\in\omega}$ of finite subsets of ω in E, the union $\bigcup\{x_i \mid i \in \omega\}$ is in E. Clearly, this notation is weaker than the "closed under chain" notion. There are sets E such that E is closed under finite chain but E is not closed under chain. Let E be the set $\{b_i \cup a \mid i \in \omega\}$ where $a = \{2k \mid k \in \omega\}$ and $b_i = \{0, \ldots, i-1, i\}$. E is obviously closed under finite chain because E does not contain any finite set of integers. E is not closed under chain because ω, being the lub of the strictly increasing chain $\{b_i \cup a\}_{i\in\omega}$, is not in E.

E has *semi-finitary property* if for every x in E, there exists some finite subset e of x such that the set $\{e_i \mid e \subseteq e_i \subseteq x\}$ is contained in E. It is clear that if E has finitary property, then E also has semi-finitary property. There are sets E such that E has semi-finitary property but E does not have finitary property. Let A be the set $\{\omega\backslash\{i\} \mid i \in \omega\}$, and let E be A^c. E obviously has semi-finitary property because all finite subsets of ω are in E. However, E does not have finitary property, for otherwise we can find some e_k such that the set $\{x \mid e_k \subseteq x\}$ is contained in E—this is clearly false by our definition of E.

Proposition 7. *If E has semi-finitary property, then E^c is closed under finite chain.*

Proof. Suppose E has semi-finitary property but E^c is not closed under finite chain. Thus we can find some increasing sequence $\{x_i\}_{i\in\omega}$ of finite subsets of ω in E^c such that $x = \bigcup\{x_i \mid i \in \omega\}$ is in E. Since E has semi-finitary property, we can find some e_k such that the set $\{e_s \mid e_k \subseteq e_s \subseteq x\}$ is contained in E. Now $e_k \subseteq x$ implies $e_k \subseteq x_i$ for some i, hence $x_i \in E$. Contradiction.

Proposition 8. *If E is closed under finite chain, then E^c has semi-finitary property.*

Proof. Suppose E is closed under finite chain but E^c does not have semi-finitary property. Let x be some element in E^c contradicting the semi-finitary property, i.e., for each finite subset e of x in E^c, there exists some e_k in E such that $e \subseteq e_k \subseteq x$ holds. Using induction, we define two increasing sequences $\{x_i\}_{i\in\omega}$ and $\{y_i\}_{i\in\omega}$ of finite subsets of ω in E^c and E respectively as follows:

$i = 0$. Write x as the union of some increasing sequence of initial segments of x. If all the initial segments of x belong to E, then x is in E since E is closed under finite chain. As x is in E^c, we can find some initial segment x_0 of x in E^c. Since x contradicts the semi-finitary property, there exists some finite subset y_0 of ω in E such that $x_0 \subseteq y_0 \subseteq x$ holds.

$i = k+1$.

Case 1. Suppose y_k has been defined and is a finite subset of x in E. Now write x as the union of some increasing sequence of initial segments z of x such that each z contains y_k. If all these z's are in E, then x would belong to E as E is closed under finite chain. As x is in E^c, we can find some initial segment z of x in E^c such that $y_k \subseteq z$ holds. Let x_{k+1} be any such z.

Case 2. Suppose x_k has been defined and is a finite subset of x in E^c. Since x contradicts the semi-finitary property, there exists some finite subset y_{k+1} of ω in $E \cap \{e_j \mid x_k \subseteq e_j \subseteq x\}$.

It is clear from our construction that x is equal to the lub of $\{x_i\}_{i\in\omega}$ as well as the lub of $\{y_i\}_{i\in\omega}$. Since E is closed under finite chain and the sequence $\{y_i\}_{i\in\omega}$ is in E, x must belong to E, contradiction.

Combining Propositions 7 and 8, we conclude:

Theorem 3. *E is closed under finite chain if and only if E^c has semi-finitary property.*

From our proof of Theorem 2, we deduce:

Theorem 4. *E is $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ if and only if E and E^c are closed under finite chain.*

Proof. ($\Rightarrow$) Trivial.

($\Leftarrow$) In our proof of Theorem 2, we only use the fact that E and E^c are closed under finite chain to show that E is $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$.

Given a subset x of ω, let $\bar{x}$ denote the real number $0.\, x(0)\ x(1) \cdots x(i) \cdots$ where for $i \in \omega$,

$$x(i) = \begin{cases} 1 & \text{if } i \in \omega, \\ 0 & \text{if otherwise.} \end{cases}$$

E is *closed under strong chain* if whenever $\{x_i\}_{i\in\omega}$ is sequence in E such that $\bar{x}_i \leqslant \bar{x}_{i+1}$ holds for all i and

$$\lim_{i\in\omega} \bar{x}_i = \overline{\bigcup\{x_i \mid i\in\omega\}},$$

then $\bigcup\{x_i \mid i\in\omega\}$ belongs to E.

It is clear that if E is closed under strong chain, then E is closed under chain. There are sets E such that E is closed under chain but E is not closed under strong chain. Let E be the set $\{a_i \mid i\in\omega\}$ where $a_i = \omega\backslash\{i\}$. E is clearly closed under chain because there is no strictly increasing chain in E. We claim that E is not closed under strong chain. To see this, note that for every $i\in\omega$, $a_i(j)=1$ for all j except $j=i$. It follows that $\bar{a}_i$ is less than $\bar{a}_{i+1}$ for all $i\in\omega$. Now

$$\lim_{i\in\omega} \overline{a_i} = 0.111\cdots = \bar{\omega} = \overline{\bigcup\{a_i \mid i\in\omega\}}.$$

But ω is not in E.

Proposition 9. *If E has finitary property, then E^c is closed under strong chain.*

Proof. Suppose E has finitary property and $\{x_i\}_{i\in\omega}$ is a sequence in E^c satisfying $\bar{x}_i \leqslant \bar{x}_{i+1}$ for all i and

$$\lim_{i\in\omega} \bar{x}_i = \overline{\bigcup\{x_i \mid i\in\omega\}}.$$

Assume $x = \bigcup\{x_i \mid i\in\omega\}$ is in E. By the finitary property of E, we can find some finite subset e of x such that the set $\{y \mid e\subseteq y\subseteq x\}$ is contained in E. Now

$$\bar{e} \leqslant \lim_{i\in\omega} \bar{x}_i = \bar{x}.$$

Since e is a finite set, there exists some $j\in\omega$ such that $e\subseteq x_i$ for all $j\leqslant i$. But this contradicts our assumption that all the x_i's are in E^c.

Proposition 10. *If E is closed under strong chain, then E^c has finitary property.*

Proof. Suppose E is closed under strong chain but E^c does not have finitary property. Hence there exists some $x\in E^c$ such that for every finite subset e of x in E^c, we can find some y in $E\cap\{y \mid e\subseteq y\subseteq x\}$. Using induction, we will construct a sequence $\{x_i\}_{i\in\omega}$ in E with the following properties:

(i) $\bar{x}_i \leqslant \bar{x}_{i+1}$ for all i;
(ii) $x = \bigcup\{x_i \mid i\in\omega\}$; and
(iii) $\lim_{i\in\omega} \bar{x}_i = \bar{x}$.

At the same time, we will define a sequence $\{y_i\}_{i\in\omega}$ with the following properties:

(i) y_i is a finite subset of x_i for every i; and
(ii) $x = \bigcup\{y_i \mid i\in\omega\}$.

i = 0. Set $y_0 = \{\text{least } j \text{ in } x\}$ and let x_0 be any element in E satisfying $y_0 \subseteq x_0 \subseteq x$—the existence of x_0 follows from the fact that x contradicts the finitary property.

i + 1. Assume $\{x_j\}_{j \leq i}$ and $\{y_j\}_{j \leq i}$ have been defined and satisfy the required properties. Since x_i belongs to E, x_i is different from x. Therefore we can find some $k \in \omega$ such that $k \in x \backslash x_i$ and for all $j < k$, $j \in x_i$ if and only if $j \in x$. Now set

$$y_{i+1} = \{k\} \cup \{j \in \omega \mid j < k \text{ and } j \in x_i\}.$$

In other words, the elements in y_{i+1} are all the elements in x which are less than or equal to k. Let x_{i+1} be any element in E satisfying $y_{i+1} \subseteq x_{i+1} \subseteq x$—$x_{i+1}$ exists because x contradicts the finitary property.

We now verify that $\{x_i\}_{i \in \omega}$ and $\{y_i\}_{i \in \omega}$ satisfy the stated properties. All properties except (i) for $\{x_i\}_{i \in \omega}$ are clear from our construction. To see $\bar{x}_i \leq \bar{x}_{i+1}$ for all i, note that k (as defined in our definition of y_{i+1}) belongs to x_{i+1} but k does not belong to x_i, and for all $j < k$, $j \in x_i$ if and only if $j \in x_{i+1}$. Since E is closed under strong chain, x belongs to E, contradiction.

Combining Propositions 9 and 10, we conclude:

Theorem 5. *E is closed under strong chain if and only if E^c has finitary property.*

A corollary of Theorems 2 and 5 is:

Corollary 1. *E is $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ if and only if E and E^c are closed under strong chain.*

6. $\mathscr{B}$ sets in Pω

The $\mathscr{B}$ sets in Pω form a subclass of the $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ sets. In this section, we give a criteria to determine when a $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ set is a $\mathscr{B}$ set. Then using our criteria, we give an example of a $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ set which is not a $\mathscr{B}$ set.

Lemma 1 tells us that the $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ sets in Pω cannot admit infinite alternating chains. Let us now introduce the notion of an alternating chain of finite length.

We say that a subset *E admits an alternating chain* $\{x_i\}_{i=0}^n$ *of length* $n+1$ if $\{x_i\}_{i=0}^n$ is an increasing chain, and for $i \leq n$, $x_i \in E \leftrightarrow i$ is even. First, we make the following observations:

(1) If E does not admit any alternating chain of length $n+1$, then E cannot admit any alternating chain of length larger than $n+1$.

(2) If E admits an infinite alternating chain (see Lemma 1), then E admits an alternating chain of arbitrary finite length.

(3) If E is closed under superset (like the open sets in Pω), then E cannot admit any alternating chain of length 2.

(4) Say that E is convex if for every x, y in E, $x \subseteq z \subseteq y$ implies $z \in E$. Then E is convex if and only if E does not admit any alternating chain of length 3. Note that all $\mathcal{J} \dot{\cap} \mathcal{S}$ sets are convex.

Proposition 11. *If E is a $\mathcal{B}$ set, then there exists some $n \in \omega$ such that E cannot admit alternating chains of length $m+1$ for all $n \leq m$.*

Proof. We mentioned earlier that the $\mathcal{B}$ sets in Pω are all the finite unions of $\mathcal{J} \dot{\cap} \mathcal{S}$ sets. If E is a $\mathcal{J} \dot{\cap} \mathcal{S}$ set, then by observation (4) above, E cannot admit any alternating chain of length greater than or equal to 3. So assume that E is a union of n $\mathcal{J} \dot{\cap} \mathcal{S}$ sets, say $E = \bigcup_{1 \leq i \leq n} E_i$ where each E_i is a $\mathcal{J} \dot{\cap} \mathcal{S}$ set. We claim that E cannot admit any alternating chain of length $2n+1$. For otherwise, one of the E_i's must admit an alternating chain of length 3 (by the pigeon-hole principle), contradicting our assumption that every E_i is a $\mathcal{J} \dot{\cap} \mathcal{S}$ set.

Proposition 12. *Suppose E is $\mathcal{B}_\sigma \cap \mathcal{B}_\delta$ but not a $\mathcal{B}$ set, then E can admit alternating chains of arbitrary finite length.*

Proof. Given any integer n, we have to show that E admits an alternating chain of length $n+1$. Without loss of generality, assume that n is even. Since E is not a $\mathcal{B}$ set, we can immediately infer that for all $m \in \omega$, $P_m^E \backslash P_{m+1}^E$ and $Q_m^E \backslash Q_{m+1}^E$ are non-empty $\mathcal{J} \dot{\cap} \mathcal{S}$ sets. Using induction, we shall define a chain $\{x_j\}_{j=0}^n$ with the following properties:

(i) $x_0 \in P_k^E \backslash P_{k+1}^E$ where k is $\frac{1}{2}n$; and

(ii)

$$x_{2j}(2j \leq n) \in P_{k-j}^E \backslash P_{k-j+1}^E,$$

$$x_{2j+1}(2j+1 \leq n) \in Q_{k-j}^E \backslash Q_{k-j+1}^E.$$

j = 0. Let x_0 be any non-empty finite subset of ω in the non-empty $\mathcal{J} \dot{\cap} \mathcal{S}$ set $P_k^E \backslash P_{k+1}^E$.

2j + 1 $(2j+1 \leq n)$. Assume x_{2j} has been defined and belongs to $P_{k-j}^E \backslash P_{k-j+1}^E$. Since x_{2j} belongs to the closure of Q_{k-j}^E, we can apply Lemma 2 to obtain some finite subset x_{2j+1} of ω in Q_{k-j}^E such that x_{2j} is a subset of x_{2j+1}. Note that x_{2j+1} cannot be in Q_{k-j+1}^E since $P_{k-j+1}^E \cup Q_{k-j+1}^E$ is a closed set and $x_{2j+1} \in Q_{k-j+1}^E$ would mean that x_{2j} is in P_{k-j+1}^E.

2j $(2j \leq n)$. Use a similar argument in the case for $2j+1$. E admits the alternating chain $\{x_i\}_{i=0}^n$. Finally we note that all the x_i's in our construction are finite sets of integers.

Remark. (1) It follows from our proof of Proposition 12 that if E is $\mathcal{B}_\sigma \cap \mathcal{B}_\delta$ and $P_n^E \neq \emptyset$, then E admits an alternating chain of length $2n+1$.

(2) Using the same technique in our proof of Proposition 12, we can establish the following fact: If E is $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ and P^E_ω is non-empty, then given any e_j in P^E_ω and any $n \in \omega$, there exists an alternating chain $\{x_i\}_{i=0}^n$ of finite subsets of ω such that x_0 is e_j and E admits $\{x_i\}_{i=0}^n$; consequently E and E^c contains infinitely many e_k's containing e_j.

Combining Propositions 11 and 12, we have the following criteria to determine when a given $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ set is a $\mathscr{B}$ set:

Theorem 6. *Suppose E is $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$, then E is a $\mathscr{B}$ set if and only if there exists some integer n such that E cannot admit any alternating chain of length $n+1$.*

We close this section by giving a $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ set C which is not a $\mathscr{B}$ set. We define C to be $\bigcup_{1 \leq n \in \omega} C_n$ where

$$C_n = \{\{n^2\}, \ldots, \{n^2, \ldots, m, \ldots, n^2+2i\}, \ldots\}$$

where $n^2 \leq m \leq n^2+2i$ and $1 \leq i \leq n$. The following properties of C show that C is $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ but not a $\mathscr{B}$ set:

(1) C is $\mathscr{B}_\sigma$ because all the elements in C are finite subsets of ω.

(2) Each C_n is an ascending chain of length $n+1$, which is part of some alternating chain of length $2n+1$ alternating in and out of C_n. It follows that C can admit an alternating chain of any finite length, hence cannot be a $\mathscr{B}$ set by Proposition 11.

(3) C is closed under chain because there is not any strictly increasing infinite chain in C. C^c is also closed under chain because C^c contains all the infinite subsets of ω. Hence by Theorem 2, C is a $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ set.

(4) Given any x in C, there exist only finitely many y's in C containing x.

(5) We claim that P^c_ω is empty. Otherwise, pick any e_i in P^c_ω. By Remark (2) after Proposition 12, there must exist infinitely many e_j's in C containing e_i, contradicting (4) above. Note that Q^c_ω is non-empty because $\emptyset \in \bigcap_{n\in\omega} Q^c_n = Q^c_\omega$. Thus Q^c_ω is a non-empty closed set and C is in $\mathscr{D}_\omega$ in the Hausdorff difference hierarchy.

7. Retracts and $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ sets : first application

Given a separable continuous lattice D, when can D be embedded as a $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ subset of $P\omega$? We will apply our knowledge of chain properties to answer this question.

Proposition 13. *Suppose D is a countable continuous lattice, then D can be embedded as a $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ subset of* Pω *if and only if the set D is finite.*

Proof. ($\Rightarrow$) Suppose D is isomorphic to some $\mathcal{B}_\sigma \cap \mathcal{B}_\delta$ subset E of $P\omega$ and D is infinite, then the "top" element of D must be mapped via isomorphism to some infinite subset y of ω in E. Since E has finitary property, we can find some finite subset e of y such that the set $\{z \mid e \subseteq z \subseteq y\}$ is contained in E. But the set $\{z \mid e \subseteq z \subseteq y\}$ is uncountable, so must E and D, contradiction.

($\Leftarrow$) Suppose the set D is $\{x_1, \ldots, x_k\}$. Consider the following embedding map i:

$$i(y) = \{j \mid x_j \sqsubseteq y\}.$$

$i(D)$ is isomorphic to D, and is a $\mathcal{B}$ set in $P\omega$ because every $i(y)$ is a finite set.

If we drop off the "countable" assumption, then Proposition 13 is false—for take D to be $P\omega$. Indeed, the following Proposition holds:

Proposition 14. *If D is an uncountable separable continuous lattice and D can be embedded as a $\mathcal{B}_\sigma \cap \mathcal{B}_\delta$ subset of* $P\omega$, *then* $P\omega$ *is a retract of D.*

Proof. Suppose D is isomorphic to some $\mathcal{B}_\sigma \cap \mathcal{B}_\delta$ subset E of $P\omega$. Take any infinite subset y of ω in E. By the finitary property of D, we can find some finite subset e of y such that the set $\{z \mid e \subseteq z \subseteq y\}$ is contained in E. But the set $\{z \mid e \subseteq z \subseteq y\}$ when partially ordered by set inclusion is isomorphic to $P\omega$. Hence $P\omega$ is a subspace and therefore a retract of D.

We conclude that if a given separable continuous lattice D can be embedded as a $\mathcal{B}_\sigma \cap \mathcal{B}_\delta$ subset of $P\omega$, then D is either finite or a universal domain.

Our proof of Proposition 13 is based on the following observation: if E has finitary property and contains some infinite set of integers, then E is uncountable; furthermore, there are infinitely many e_i's in E.

Exploiting this observation furthermore, we show below that the converse of Proposition 14 is false.

Proposition 15. *There exists a universal domain D such that D cannot be embedded as a $\mathcal{B}_\sigma \cap \mathcal{B}_\delta$ subset of* $P\omega$.

Proof. Let D be $[P\omega \to P\omega]$ and i any map which embeds D in $P\omega$. Note that $P\omega$ is a retract of D. We want to show that $i(D)$ cannot be a $\mathcal{B}_\sigma \cap \mathcal{B}_\delta$ subset of $P\omega$. It suffices to show that there exists at most one e_i in $i(D)$.

Claim. For every $f \in [P\omega \to P\omega]$, $f = \perp$ or f is "infinite" in the sense that there exist infinitely many g's $\sqsubseteq f$ in $[P\omega \to P\omega]$.

Proof. It suffices to show that every step function (except $\perp$) in $[P\omega \to P\omega]$ is infinite because every continuous function is given by the lub of a sequence of step functions.

So consider a simple step function like $\bar{e}(\{0\},\{1\})$. Recall that for $x \in P\omega$,

$$\bar{e}(\{0\},\{1\})(x) = \begin{cases} \{1\} & \text{if } 0 \in x, \\ \emptyset & \text{if otherwise.} \end{cases}$$

For any e_k containing 0, it is clear that $\bar{e}(e_k,\{1\})$ is less than $\bar{e}(\{0\},\{1\})$ in the partial ordering of $[P\omega \to P\omega]$. Since there are infinitely many e_k's containing 0, we conclude that $\bar{e}(\{0\},\{1\})$ is infinite.

Since i is an embedding map, i must map all the "infinite" elements in $[P\omega \to P\omega]$ to infinite sets of integers in $P\omega$. And since $\perp$ is the only non-infinite element in $[P\omega \to P\omega]$, $i(D)$ can contain at most one finite set of integers.

The claim in Proposition 15 essentially says that there is no non-trivial minimal element in $[P\omega \to P\omega]$ with respect to the pointwise ordering. This is why $[P\omega \to P\omega]$ is not isomorphic to $P\omega$.

8. Effective open sets: second application

An open set in $P\omega$ is some countable union of basic neighborhoods which are of the form $\{x \mid e_i \subseteq x\}$. It is said to be effective if it is some effective union of basic neighborhoods. More formally, E is an *effective open set* if there exists some recursive function $f: \omega \to \omega$ such that

$$E = \bigcup_{i \in \omega} \{x \mid e_{f(i)} \subseteq x\}.$$

Equivalently, E is effective if and only if the set $\{n \mid e_n \in E\}$ is r.e., that is, we can enumerate the indices of all the finite sets in E.

If an open set E is effective, then it is not hard to see that we can as well enumerate the indices of all the r.e. sets (or computable elements [8]) in E because:

$$W_n \in E \leftrightarrow (\exists i)(e_{f(i)} \subseteq W_n)$$

(where W_n's are as in [4]) and the predicate $e_{f(i)} \subseteq W_n$ is r.e. in $f(i)$ and n. It turns out that if E is $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ and the set $\{n \mid W_n \in E\}$ is r.e., then E is an effective open set. Our proof of the latter fact rests on our previous observations and the following version of Rice–Shapiro Theorem [4]:

Rice–Shapiro Theorem. *Let $\mathscr{C}$ be any non-empty collection of* r.e. *sets. If the set $\{n \mid W_n \in \mathscr{C}\}$ is* r.e., *then there exists a recursive $f: \omega \to \omega$ such that*:

$$W_n \in \mathscr{C} \leftrightarrow (\exists i)(e_{f(i)} \subseteq W_n).$$

Theorem 7. E *is an effective open set in* Pω *if and only if* E *is* $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ *and the set* $\{n \mid W_n \in E\}$ *is* r.e.

Proof. (⇒) Trivial.

(⇐) Suppose E is a non-empty $\mathscr{B}_\sigma \cap \mathscr{B}_\delta$ set and the set $\{n \mid W_n \in E\}$ is r.e., we need to show that E is an effective open set. Let $\mathscr{C}$ be the collection of r.e. sets in E. Since E is a $\mathscr{B}_\sigma$ set, we know by Proposition 1 that $\mathscr{C}$ is non-empty. Therefore applying Rice–Shapiro Theorem, we get some recursive f such that

$$W_n \in \mathscr{C} \leftrightarrow (\exists i)(e_{f(i)} \subseteq W_n).$$

It suffices to prove the following claim:

Claim. $E = \bigcup_{i \in \omega} \{x \mid e_{f(i)} \subseteq x\}$.

Proof. Suppose $x \in E$. Because E has finitary property, there exists some finite subset e of x such that the set $\{y \mid e \subseteq y \subseteq x\}$ is contained in E. But e is a finite set, hence $e = W_n$ for some n. By Rice–Shapiro Theorem, $e_{f(i)} \subseteq W_n$ for some i. Therefore, we have

$$e_{f(i)} \subseteq W_n \subseteq x.$$

Conversely, suppose $e_{f(i)} \subseteq x$. Note:

$$x = \bigcup \{W_n \mid e_{f(i)} \subseteq W_n \subseteq x\}.$$

By Rice–Shapiro Theorem, the set $\{W_n \mid e_{f(i)} \subseteq W_n \subseteq x\}$ is contained in E. Since E is direct -complete and x is the lub of the directed set $\{W_n \mid e_{f(i)} \subseteq W_n \subseteq x\}$, we conclude that x belongs to E.

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